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A NOTE ON TOPOLOGICAL n-GROUPS

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Abstract. In the present paper is proved the following proposition. Let (Q, A) be an n-group, $^{-1}$ its inversing operation, $n \geq 2$ and Q is equipped with a topology $\mathcal{O}$. Also let

$$\begin{aligned} {}^{-1}A\left(x, a_1^{n-2}, y\right) &= z \overset{def}{\Longleftrightarrow} A\left(z, a_1^{n-2}, y\right) = x \text{ and} \\ A^{-1}\left(x, a_1^{n-2}, y\right) &= z \overset{def}{\Longleftrightarrow} A\left(x, a_1^{n-2}, z\right) = y \end{aligned}$$

for all $x, y, z \in Q$ and for every sequence a_1^{n-2} over Q. Then the following statements are equivalent: (i) the n-ary operation A is continuous in $\mathcal{O}$ and the $(n-1)$-ary operation $^{-1}$ is continuous in $\mathcal{O}$; (ii) the n-ary operation ^{-1}A is continuous in $\mathcal{O}$; and (iii) the n-ary operatin A^{-1} is continuous in $\mathcal{O}$. / See, also Remark 2.2. /

1. Preliminaries

1.1. Definition: Let $n \geq 2$ and let (Q, A) be an n-groupoid. We say that (Q, A) is a Dörnte n-group / briefly: n-group / iff is an n-semigroup and an n-quasigroup as well.*

1.2. Proposition [14]: *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then the following statements are equivalent : (i) (Q, A) is an n-group; (ii) there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A, {}^{-1}, \mathbf{e}\})$ / of the type $\langle n, n-1, n-2\rangle$ /.*

(a) $$A\left(x_1^{n-2}, A\left(x_{n-1}^{2n-2}\right), x_{2n-1}\right) = A\left(x_1^{n-1}, A\left(x_n^{2n-1}\right)\right),$$

(b) $$A\left(\mathbf{e}\left(a_1^{n-2}\right), a_1^{n-2}, x\right) = x \quad \textit{and}$$

(c) $$A\left(\left(a_1^{n-2}, a\right)^{-1}, a_1^{n-2}, a\right) = \mathbf{e}\left(a_1^{n-2}\right); \quad \textit{and}$$

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*A notion of an n-group was introduced by W. Dörnte in [1] as a generalization of the notion of a group. See, also [2-4].

(iii) there are mappings $^{-1}$ *and* $\mathbf{e}$ *respectively of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q,\{A,\ ^{-1},\mathbf{e}\})$ *[of the type* $\langle n,n-1,n-2\rangle$ *]*

$$(\bar{a})\qquad A\left(A\left(x_1^n\right),x_{n+1}^{2n-1}\right)=A\left(x_1,A\left(x_2^{n+1}\right),x_{n+2}^{2n-1}\right),$$

$$(\bar{b})\qquad A\left(x,a_1^{n-2},\mathbf{e}\left(a_1^{n-2}\right)\right)=x$$

$$(\bar{c})\qquad A\left(a,a_1^{n-2},\left(a_1^{n-2},a\right)^{-1}\right)=\mathbf{e}\left(a_1^{n-2}\right).$$

1.3. Remarks: $\mathbf{e}$ is an $\{1,n\}$-neutral operation of n-groupoid (Q,A) iff algebra $(Q,\{A,\mathbf{e}\})$ of type $\langle n,n-2\rangle$ satisfies the laws (b) and $(\bar{b})$ from 1.2 /:[11] /. The notion of $\{i,j\}$-neutral operation $(i,j\in\{1,...,n\},\ i<j)$ of an n-groupoid is defined in a similar way / :[11] /. Every n-groupoid has at most one $\{i,j\}$-neutral operation / :[11] /. In every n-group $(n\geq 2)$ there is an $\{1,n\}$-neutral operation / :[11] /. There are n-groups without $\{i,j\}$-neutral operation with $\{i,j\}\neq\{1,n\}$ / :[13] /. In [13], n-groups with $\{i,j\}$-neutral operations, for $\{i,j\}\neq\{1,n\}$ are described. Operation $^{-1}$ from 1.2/ $(c),(\bar{c})$/ is a generalization of the inversing operation in a group. In fact, if (Q,A) is an n-group, $n\geq 2$, then for every $a\in Q$ and for every sequence a_1^{n-2} over Q is

$$(a_1^{n-2},a)^{-1}\stackrel{def}{=}\mathsf{E}(a_1^{n-2},a,a_1^{n-2}),$$

where E is an $\{1,2n-1\}$-neutral operation of the $(2n-1)$-group $(Q,\overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1})\stackrel{def}{=}A(A(x_1^n),x_{n+1}^{2n-1})$/ :[12] /. (For $n=2, a^{-1}=\mathsf{E}(a); a^{-1}$ is the inverse element of the element a with respect to the neutral element $\mathbf{e}(\emptyset)$ of the group (Q,A).)

2. Result

2.1. Theorem: *Let* (Q,A) *be an n-group,* $^{-1}$ *its inversing operation [: [12,14], 1.3],* $n\geq 2$ *and* Q *is equipped with a topology* $\mathcal{O}$. *Also let*

$$(0)\qquad {}^{-1}A\left(x,a_1^{n-2},y\right)=z\stackrel{def}{\Longleftrightarrow}A\left(z,a_1^{n-2},y\right)=x\ \text{ and}$$

$$(\bar{0})\qquad A^{-1}\left(x,a_1^{n-2},y\right)=z\stackrel{def}{\Longleftrightarrow}A\left(x,a_1^{n-2},z\right)=y$$

for all $x,y,z\in Q$ *and for every sequence* a_1^{n-2} *over* Q. *Then the following statements are equivalent:*

(i) the n-ary operation A *is continuous in* $\mathcal{O}$ *and the* $(n-1)$*-ary operation* $^{-1}$ *is continuous in* $\mathcal{O}$*;*

(ii) the n-ary operation ^{-1}A *is continuous in* $\mathcal{O}$*; and*

(iii) the n-ary operation A^{-1} *is a continuous in* $\mathcal{O}$.

Proof. 1) Let (Q, A) be an n-group, $^{-1}$ its inversing operation, $\mathbf{e}$ its $\{1, n\}$-neutral operation and $n \geq 2$. Also let

$$\begin{aligned} {}^{-1}A\left(x, a_1^{n-2}, y\right) &= z \overset{def}{\Longleftrightarrow} A\left(z, a_1^{n-2}, y\right) = x \text{ and} \\ A^{-1}\left(x, a_1^{n-2}, y\right) &= z \overset{def}{\Longleftrightarrow} A\left(x, a_1^{n-2}, z\right) = y \end{aligned}$$

for all $x, y, z \in Q$ and for every sequence a_1^{n-2} over Q. Then, for all $x, y \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold

1° $${}^{-1}A\left(x, a_1^{n-2}, y\right) = A\left(x, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right),$$

2° $$\mathbf{e}\left(a_1^{n-2}\right) = {}^{-1}A\left(x, a_1^{n-2}, x\right),$$

3° $$\left(a_1^{n-2}, x\right)^{-1} = {}^{-1}A\left({}^{-1}A\left(x, a_1^{n-2}, x\right), a_1^{n-2}, x\right) \text{ and}$$

4° $$A\left(x, a_1^{n-2}, y\right) = {}^{-1}A\left(x, a_1^{n-2}, {}^{-1}A\left({}^{-1}A\left(y, a_1^{n-2}, y\right), a_1^{n-2}, y\right)\right).$$

Sketch of the proof of 1°:

$$\begin{gathered} {}^{-1}A\left(x, a_1^{n-2}, y\right) = z \Leftrightarrow A\left(z, a_1^{n-2}, y\right) = x \Leftrightarrow \\ A\left(A\left(z, a_1^{n-2}, y\right), a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right) = A\left(x, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right) \Leftrightarrow \\ A\left(z, a_1^{n-2}, A\left(y, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right)\right) = A\left(x, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right) \Leftrightarrow \\ A\left(z, a_1^{n-2}, \mathbf{e}\left(a_1^{n-2}\right)\right) = A\left(x, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right) \Leftrightarrow \\ z = A\left(x, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right) \end{gathered}$$

/: (0), 1.1, 1.2, 1.3 /.

Sketch of the proof of 2°:

$${}^{-1}A\left(x, a_1^{n-2}, x\right) = \mathbf{e}\left(a_1^{n-2}\right) \Leftrightarrow A\left(\mathbf{e}\left(a_1^{n-2}\right), a_1^{n-2}, x\right) = x \text{ /: (0) /, 1.2, 1.3.}$$

Sketch of the proof of 3°:

$$\begin{gathered} {}^{-1}A\left({}^{-1}A\left(x, a_1^{n-2}, x\right), a_1^{n-2}, x\right) = \left(a_1^{n-2}, x\right)^{-1} \Leftrightarrow \\ A\left(\left(a_1^{n-2}, x\right)^{-1}, a_1^{n-2}, x\right) = {}^{-1}A\left(x, a_1^{n-2}, x\right) \Leftrightarrow \\ A\left(\left(a_1^{n-2}, x\right)^{-1}, a_1^{n-2}, x\right) = \mathbf{e}\left(a_1^{n-2}\right) \end{gathered}$$

/: (0), 2° /, 1.2, 1.3.

Sketch of the proof of 4°:

$$\begin{gathered} A\left(x, a_1^{n-2}, y\right) = {}^{-1}A\left(x, a_1^{n-2}, {}^{-1}A\left({}^{-1}A\left(y, a_1^{n-2}, y\right), a_1^{n-2}, y\right)\right) \Leftrightarrow \\ x = A\left(A\left(x, a_1^{n-2}, y\right), a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right) \Leftrightarrow \\ x = A\left(x, a_1^{n-2}, A\left(y, a_1^{n-2}, \left(a_1^{n-2}, y\right)^{-1}\right)\right) \Leftrightarrow \\ x = A\left(x, a_1^{n-2}, \mathbf{e}\left(a_1^{n-2}\right)\right) \end{gathered}$$

/: (0), 3°, 1.1, 1.2 /, 1.2, 1.3.

Similarly, it is possible to prove also the following equalities

°1 $$A^{-1}\left(x,a_1^{n-2},y\right)=A\left(\left(a_1^{n-2},x\right)^{-1},a_1^{n-2},y\right),$$

°2 $$\mathbf{e}\left(a_1^{n-2}\right)=A^{-1}\left(x,a_1^{n-2},x\right),$$

°3 $$\left(a_1^{n-2},x\right)^{-1}=A^{-1}\left(x,a_1^{n-2},A^{-1}\left(x,a_1^{n-2},x\right)\right) \text{ and}$$

°4 $$A\left(x,a_1^{n-2},y\right)=A^{-1}\left(A^{-1}\left(x,a_1^{n-2},A^{-1}\left(x,a_1^{n-2},x\right)\right),a_1^{n-2},y\right)$$

for all $x,y\in Q$ and for every sequence a_1^{n-2} over Q.

2) Let Q be equipped with topology $\mathcal{O}$. Also let the n-ary operation A is continuous in $\mathcal{O}$ and the $(n-1)$-ary operation ${}^{-1}$ is continuous in $\mathcal{O}$. Then, by 1° (°1), we conclude that the n-ary operation ${}^{-1}\!A$ (A^{-1}) is continuous in $\mathcal{O}$.

3) Let Q be equipped with topology $\mathcal{O}$. Also let the n-ary operation ${}^{-1}\!A$ (A^{-1}) is continuous in $\mathcal{O}$.Then, by 3° and 4° (°3 and °4), we conclude that the n-ary operation A is continuous in $\mathcal{O}$ and the $(n-1)$-ary operation ${}^{-1}$ is continuous in $\mathcal{O}$. □

2.2. Remark: Topological n-groups have been defined in mutually different ways. The definitions in [6], [10] and [15] are related to each $n\geq 2$, while those in [7] and [8] are restricted (only) to $n\geq 3$. All definitions from the cited papers are mutually equivalent for $n\geq 3$, while the definitions from the papers [6], [10] and [15] are also mutually equivalent for $n=2$ *[* : [9], [15] *]*. (Definition of the topological n-groups from [10] is the definition 2.3 from [9].) Topological n-group is defined in [15] by (i) from 2.1. S.A. Rusakov has proved (1984) that the n-group (Q,A) is topological with respect to the topology $\mathcal{O}$ iff A and ${}^{-1}\!A$ or A and A^{-1} are continuous in the topology $\mathcal{O}$ *[* : see [9] *]*. Assertion 2.1 is well-known for $n=2$ *[* : e.g., [5], p. 105; ${}^{-1}\!A(x,y)=A(x,y^{-1})$ *]*. □

3. References

[1] W. Dörnte: *Untersuchengen über einen verallgemeinerten Gruppenbegriff*, Math. Z., **29**(1928),1-19.

[2] R. H. Bruck: *A survey of binary systems*, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[3] V. D Belousov: *n−ary quasigroups*, "Stiinca", Kishinev 1972. (In Russian.)

[4] A. G. Kurosh: *General algebra* (lectures 1969-1970), "Nauka", Moscow 1974. (In Russian.)

[5] L. S. Pontryagin: *Topological groups*, "Nauka", Moscow 1973. (In Russian.)

[6] Ǵ. Čupona: *On topological n-groups*, Bilten na Društ. na mat. i fiz. od SRM, **22**(1971), 5-10.

[7] G. Crombez, G. Six: *On topological n-groups*, Abh. Math. Sem. Hamburg **41**(1974), 115-124.

[8] R. M. Žižović: *Topological analogi of Hosszú-Gluskin's theorem*, Matematički vesnik, **13(28)**(1976), 233-235. (In Serbo-Croation).

[9] S. A. Rusakov: *Algebraic n-ary Systems*, "Navuka i Tehnika", Minsk 1992. (In Russian.)

[10] N. Enders: *On topological n-groups and their corresponding groups*, Discussiones Mathematicae, Algebra and Stochastic Methods, **15**(1995),163-169.

[11] J. Ušan: *Neutral operations of n−groupoids*, Rev. of Research, Fac.of Sci. Univ. of Novi Sad, Math. Ser., **18-2**(1988), 117-126. (In Russian.)

[12] J. Ušan: *A comment on n-groups*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser., **24-1**(1994), 281-288.

[13] J. Ušan: *On n-groups with* $\{i,j\}$*-neutral operation for* $\{i,j\} \neq \{1,n\}$, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser., **25-2**(1995), 167-178.

[14] J. Ušan: *n−groups,* $n \geq 2$*, as variety of type* $< n, n-1, n-2 >$, Algebra and Model Theory, Collection of papers edited by A. G. Pinus and K. N. Ponomaryov, Novosibirsk 1997, 182-208.

[15] J. Ušan: *On topological n-groups*, Math. Moravica, **2**(1998), 149-159.

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